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IRREDUCIBLE CONNECTED LIE SUBGROUPS OF $GL_n(R)$ ARE CLOSED

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ABSTRACT

If G is a connected real Lie group and $\pi: G \to \mathrm{Aut}(V)$ a continuous irreducible finite-dimensional real representation then we show that $\pi(G)$ is closed in $\mathrm{Aut}(V)$. A similar result is valid in the complex case.

The following theorem is quoted from Kobayashi and Nomizu [4, p. 277].

THEOREM 2. *Let G be a connected Lie subgroup of* $SO(n)$ *which acts irreducibly on* $\mathbf{R}^n$. *Then G is closed in* $SO(n)$.

We shall show that $SO(n)$ can be replaced by $GL_n(\mathbf{R})$ in the above theorem.

THEOREM A. *Let G be a connected Lie subgroup of* $GL_n(\mathbf{R})$ *which acts irreducibly on* $\mathbf{R}^n$. *Then G is closed in* $GL_n(\mathbf{R})$.

The proof is based on the following result [2, prop. 1].

PROPOSITION 1. *Let G be a real Lie group, H an analytic subgroup of G, N the radical of H and S a maximal semi-simple analytic subgroup of H. Then $N \cap S$ is contained in the center Z of S. If the index of $N \cap S$ in Z is finite, then $\bar{H} = \bar{N}S$ and $\bar{N}$ is the radical R of $\bar{H}$.*

(The operator $^{-}$ denotes the closure in G, and *analytic subgroup* is a synonym for *connected Lie subgroup.*)

PROOF OF THEOREM A. Since G acts irreducibly in $\mathbf{R}^n$ it follows that $\mathbf{R}^n$ is also simple as a module for the Lie algebra $L(G)$ of G. By [1, prop. 5, pp. 78–79.], $L(G)$ is reductive and hence we have $G = NS$ where N is the identity component of the center of G (and also the radical of G), and S is a normal semi-simple subgroup of G. By [3, prop. 4.1, p. 221], the center Z of S is finite

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and so we can apply the above Proposition 1 to $GL_n(\mathbf{R})$ and G. It follows that $\bar{G} = \bar{N}S$ where $\bar{\ }$ is the closure operator in $GL_n(\mathbf{R})$. Thus, it suffices to show that $\bar{N} = N$.

The endomorphism ring $\mathbf{D} = \mathrm{End}_G(\mathbf{R}^n)$ is a finite-dimensional real division algebra (by Schur's lemma) and so $\mathbf{D}$ is isomorphic to one of the algebras $\mathbf{R}$ (real numbers), $\mathbf{C}$ (complex numbers), or $\mathbf{H}$ (real quaternions). Since $\mathbf{D}$ is a vector subspace of the matrix algebra $M_n(\mathbf{R})$, it is closed in $M_n(\mathbf{R})$. Consequently the multiplicative group $\mathbf{D}^* = \mathbf{D}\backslash\{0\}$ is closed in $GL_n(\mathbf{R})$. Now, we note that every abelian connected Lie subgroup of $\mathbf{D}^*$ is closed in $\mathbf{D}^*$ (this is obvious if $\mathbf{D} \simeq \mathbf{R}$ or $\mathbf{C}$, and is easy to check when $\mathbf{D} \cong \mathbf{H}$). Since $N \subset \mathbf{D}^*$, it follows that N is closed in $\mathbf{D}^*$. But $\mathbf{D}^*$ is closed in $GL_n(\mathbf{R})$ and consequently $\bar{N} = N$. This completes the proof.

One can state this theorem in a slightly more general form:

THEOREM A′. *Let G be a connected real Lie group and $\pi: G \rightarrow \mathrm{Aut}(V)$ a continuous finite-dimensional real representation. If π is irreducible then $\pi(G)$ is closed in* $\mathrm{Aut}(V)$.

Indeed, it suffices to remark that $\pi(G)$ is an irreducible analytic subgroup of $\mathrm{Aut}(V)$ and that $\mathrm{Aut}(V) \cong GL_n(\mathbf{R})$, $n = \dim V$.

The complex version of Theorem A′ is also valid, i.e., we have

THEOREM B′. *Let G be a connected complex Lie group and $\pi: G \rightarrow \mathrm{Aut}(V)$ a finite-dimensional complex analytic representation. If π is irreducible then $\pi(G)$ is closed in* $\mathrm{Aut}(V)$.

It suffices to consider the case when $G = \pi(G)$. Then the proof is similar to the above proof of Theorem A and in fact simpler because now we have $\mathbf{D} \cong \mathbf{C}$.

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